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AUG 80 H BERESTYCKI, P L LIONS

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H. Berestycki and P. L. Lions

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**Mathematics Research Center
University of Wisconsin-Madison
610 Walnut Street
Madison, Wisconsin 53706**

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SEMITLINEAR ELLIPTIC PROBLEMS.

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ABSTRACT

In this paper a semilinear elliptic second-order problem is considered. Under very general assumptions we give a precise description of the number of solutions of the problem. These results extend in particular a result due to A. Ambrosetti and G. Prodi.

AMS(MOS) Subject Classification: 35J60, 35P30

Key Words: Semilinear elliptic equations, topological degree

Work Unit Number 1: Applied Analysis

^fC.N.R.S.; Laboratoire d'Analyse Numérique, Université P. et M. Curie, 4 Place Jussieu, 75230 Paris Cedex 05.

^fMathematics Research Center, University of Wisconsin-Madison, and C.N.R.S., Laboratoire d'Analyse Numérique, Université P. et M. Curie, 4 Place Jussieu, 75230 Paris Cedex 05.

SIGNIFICANCE AND EXPLANATION

Semilinear elliptic equations (that is, for example, the Laplace equation perturbed by a nonlinearity) occur in many applications, for example in combustion theory, biology, population genetics, astrophysics . . . Under general assumptions, we give a precise description of the number of solutions of the equation.

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SHARP EXISTENCE RESULTS FOR A CLASS
OF SEMILINEAR ELLIPTIC PROBLEMS

H. Berestycki[†] and P. L. Lions [‡]

Introduction.

The problem considered here is of the following type: let Ω be a bounded regular domain in \mathbb{R}^N , we look for solutions u of

$$(1) \quad -\Delta u = g(x, u) + f(x) \text{ in } \Omega, \quad u \in C^2(\bar{\Omega}), \quad \frac{\partial u}{\partial v} = 0 \text{ on } \partial\Omega;$$

where v is the unit outward normal to $\partial\Omega$, $f \in C^{0,\alpha}(\bar{\Omega})$ (f s.r some

$0 < \alpha < 1$) and $g(x, u)$ is a smooth nonlinearity satisfying essentially:

$$(2) \quad \lim_{t \rightarrow -\infty} \frac{g(x, t)}{t} < 0 < \lim_{t \rightarrow +\infty} \frac{g(x, t)}{t} \quad (\text{uniformly in } x \in \bar{\Omega});$$

and some appropriate growth condition at $+\infty$.

If $f(x) = t\varphi(x) + f_1(x)$, where $t \in \mathbb{R}$, $\varphi \in C^{0,\alpha}(\bar{\Omega})$ with

$$(3) \quad \varphi > 0 \text{ in } \bar{\Omega}, \quad \varphi \not\equiv 0$$

we prove (see Section I) that there exists $t_0 (= t_0(\varphi, f_1)) \in \mathbb{R}$ such that

- i) if $t > t_0$, there is no solution of (1);
- ii) if $t = t_0$, there is at least a minimum solution of (1);
- iii) if $t < t_0$, there is a minimum solution of (1) and there are at least two distinct solutions.

This result extends and sharpens many earlier results due to A. Ambrosetti and G. Prodi [2], M. S. Berger and E. Podolak [5], P. Hess and B. Ruf [9], J. L. Kazdan and F. W. Warner [11], H. Berestycki [4], H. Amann and P. Hess [1], E. N. Dancer [8]. The main assumption that we remove is the "at most linear

[†]C.N.R.S.; Laboratoire d'Analyse Numérique, Université P. et M. Curie, 4 Place Jussieu, 75230 Paris Cedex 05.

[‡]Mathematics Research Center, University of Wisconsin-Madison and C.N.R.S., Laboratoire d'Analyse Numérique, Université P. et M. Curie, 4 Place Jussieu, 75230 Paris Cedex 05.

growth at $+\infty$ " and in addition we prove the existence for $t < t_0$ of two ordered solutions.

In Section II, we consider the special case of $f(x,t)$ convex in t and we give some results of a geometrical nature concerning the set of functions f for which (1) admits a solution. Our main concern is to extend the results of H. Berestycki [4] to the case in which we no longer assume that g grows at most linearly at $+\infty$.

I. The general case.

Let α be in $(0,1)$ and let $f \in C^{0,\alpha}(\bar{\Omega})$. We assume that the nonlinearity $g(x,t)$ belongs to $C^{0,\alpha}(\bar{\Omega})$ (uniformly for t bounded) and $g(x,t)$ is Lipschitz continuous in t , uniformly for x in $\bar{\Omega}$. In addition, we restrict the growth of $g(x,t)$ for t large by the following assumption:

$$(4) \quad \lim_{t \rightarrow +\infty} g(x,t) t^{-p} = 0, \text{ uniformly in } x \in \bar{\Omega}, \text{ for some } p < \frac{N}{N-2};$$

(if $N = 2$, $\frac{N}{N-2}$ may be replaced by any $p < \infty$; and if $N = 1$, we make no assumption at all). We then have

Theorem I.1: Under assumptions (2), (4) and if $f(x) = t\varphi(x) + f_1(x)$ with $\varphi \in C^{0,\alpha}(\bar{\Omega})$ satisfying (3), there exists $t_0 \in \mathbb{R}$ ($t_0 = t_0(\varphi, f_1)$) such that:

- i) if $t > t_0$, there is no solution of (1);
- ii) if $t = t_0$, there is at least a minimum solution of (1);
- iii) if $t < t_0$, there is a minimum solution of (1) and there are at least two distinct solutions.

Remark I.1.: As it will be clear from an inspection of the proof, the same result holds if we replace $-\Delta$ by any uniformly elliptic second-order operator (with smooth coefficients) and if we suppose that g depends also on $\nabla u: g = g(x,u,p)$ for $(x,u,p) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$; we then need to assume that $g(x,t,p)$ is bounded for $(x,p) \in \bar{\Omega} \times \mathbb{R}^N$ and t bounded and that the limits in (3), (4) hold uniformly in $p \in \mathbb{R}^N$. In addition, we may also replace (1) by

$$(1') \quad -\Delta u = f(x,u,t) \text{ in } \Omega, u \in C^2(\bar{\Omega}), \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega;$$

assuming as in [1]:

$$(5) \quad \forall m \in \mathbb{R}, \exists \varphi \in C(\bar{\Omega}) \text{ such that } \frac{\partial f}{\partial t}(x,\xi,t) > \varphi(x) > 0, \\ \text{for } x \text{ in } \Omega, \xi > m \text{ and } t \in \mathbb{R}.$$

Remark I.2. Assumption (4) is a technical assumption which insures that solutions of (1) are a priori bounded (cf. the proof of Theorem I.1 below). We believe that the same result is true with $\frac{N}{N-2}$ replaced by $\frac{N+2}{N-2}$. For a similar reason, if we replace Neumann boundary condition by a more general one, then we need to replace $\frac{N}{N-2}$ by $\frac{N+1}{N-1}$ (we then use in the proof of Theorem I.1, the a priori estimates of H. Brezis and R. E. L. Turner [6]).

Proof of Theorem I.1: The proof is divided in several steps: we prove

- 1) there exist arbitrary negative subsolutions of (1), 2) the set of t such that (1) has a solution is of the form $(-\infty, t_0]$, 3) that (1) has always a minimum solution if $t < t_0$, and finally 4) that (1) has two distinct solutions for $t < t_0$.

a) Let $\psi \in C^{0,\alpha}(\bar{\Omega})$, then there exists $v \in C^2(\bar{\Omega})$ such that

$$-\Delta v \leq g(x,v) + f(x) \text{ in } \bar{\Omega}, \quad v \leq \psi \text{ in } \bar{\Omega}, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Indeed, because of (2) we have

$$(6) \quad g(x,t) \geq -at - c \quad \text{for } t, x \in \mathbb{R} \times \bar{\Omega} \quad \text{and for some } a, c > 0.$$

Then if we define v by $v = -\max\left(\frac{1}{a}(\|f\|_\infty + C), \|\psi\|_\infty\right)$ we have obviously $v \leq \psi$ and

$$-\Delta v = 0 \leq -a v - C + f(x) \leq g(x,v) + f(x).$$

b) we first prove that if t is bounded, all possible solutions of (1) are bounded in $C^{2,\alpha}(\bar{\Omega})$.

Indeed, because of (6), we deduce obviously from the maximum principle that

if u is a solution of (1), one has: $u(x) \geq -\frac{1}{a}(\|f\|_\infty + C)$. In particular u^+ is bounded in $L^\infty(\Omega)$. Next, if we integrate (1) on Ω , we obtain

$$\int g(x,u) = -\int f(x) + \text{Const.};$$

because u satisfies (2) and u^+ is bounded in $L^\infty(\Omega)$, this implies:

$$\int |g(x,u)| dx \leq \text{Const.}, \quad \int |f(x)| dx \leq \text{Const.}.$$

In particular we have: $\|\Delta u\|_{L^1}, \|u\|_{L^1} \leq \text{Const.}$. This implies by well-known regularity results: $\|u\|_{L^p} \leq \text{Const.}, \forall p < \frac{N}{N-2}$. Since g satisfies (4), it is easy to obtain by a bootstrap argument:

$$\|u\|_{L^\infty} \leq \text{Const.} .$$

Let us prove now that if (1) has a solution for some t , then for all $s < t$, (1) has a solution. Indeed let u be a solution of (1) for t and let $s < t$, obviously u is a supersolution of (1) (for s) i.e.:

$$-\Delta u = g(x, u) + t\varphi + f_1 \geq g(x, u) + s\varphi + f_1 .$$

On the other hand, by step 1) above, we know there exists v satisfying

$$-\Delta v \leq g(x, v) + s\varphi + f_1 , \quad v \leq u .$$

Then by classical results on sub and supersolutions, this proves our claim.

Thus we know that the set of t such that (1) has a solution is either $(-\infty, t_0]$ (with $t_0 < +\infty$) or $(-\infty, +\infty)$ (it is necessarily closed in view of the a priori bounds proved above). We just need to prove that (1) cannot have a solution for all t : we argue by contradiction and we suppose (1) has a solution u_t for all t . Then we define u_1, u_2 by

$$\begin{cases} -\Delta u_1 + \alpha u_1 = \varphi \text{ in } \Omega, \quad \frac{\partial u_1}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad u_1 \in C^2(\bar{\Omega}) \\ -\Delta u_2 + \alpha u_2 = f_1 - C \text{ in } \Omega, \quad \frac{\partial u_2}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad u_2 \in C^2(\bar{\Omega}) . \end{cases}$$

In view of (6), we have

$$u_t \geq t u_1 + u_2 \text{ in } \bar{\Omega} .$$

Since φ satisfies (3), we have $u_1 > 0$ in $\bar{\Omega}$ and thus for t large enough

$$u_t > 0 \text{ in } \bar{\Omega} .$$

Because of (2), we have: $g(x, t) \geq \alpha t - C$ for $t \geq 0$ for some $\alpha, C > 0$.

Then integrating (1) on Ω and using the fact that u_t is positive, we obtain

$$\alpha \int_{\Omega} u_t dx + t \int_{\Omega} \varphi dx \leq \text{Const. (indep. of } t \text{)};$$

since $\int_{\Omega} \varphi dx > 0$, we obtain a contradiction for t large enough.

3) Now let $t < t_0$, then (1) has always a minimum solution if $t < t_0$.

We already know that (1) has a solution u and that all possible solutions of

(1) satisfy: $u \geq -\frac{1}{\alpha} (\|f\|_{\infty} + C)$ (α, C given by (6)). But

$v = -\frac{1}{\alpha} (\|f\|_{\infty} + C)$ is a subsolution of (1) (take $\psi = 0$ in Step 1)) and

thus $u > v$. Then, by well-known results, this implies that (1) has a minimum solution \tilde{u} among all solutions satisfying: $w \geq v$ in $\bar{\Omega}$. Since all solutions w of (1) satisfy: $w \geq v$ in $\bar{\Omega}$, \tilde{u} is in fact the minimum solution of (1).

4) Finally let $t < t_0$, and let us prove that (1) has two distinct solutions. We are going to use a topological degree argument (we refer to J. Leray and J. Schauder [12], or to L. Nirenberg [15] for a definition and the main properties of the Leray-Schauder degree).

Let us first introduce some notations, let u_{t_0} be the minimum solution of (1) where f is given by $t_0\varphi + f_1$. By Steps 1), 2), 3), we know there exists a strict subsolution v of

$$-\Delta v \leq g(x, v) + t\varphi + f_1, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

and a minimum solution u_t of (1) (with f given by $t\varphi + f_1$) satisfying:

$$v < u_t < u_{t_0} \quad \text{in } \bar{\Omega}.$$

We are going to prove the existence of a solution u of (1) which does not satisfy:

$$v < u < u_{t_0} \quad \text{in } \bar{\Omega}$$

and thus $u > u_t$ in $\bar{\Omega}$, $u \neq u_{t_0}$ in $\bar{\Omega}$.

By Step 2) and the a priori bounds, we may choose $C_1 > 0$ such that all solutions u of (1) satisfy: $\|u\|_{C(\bar{\Omega})} < C_1$, and we may assume

$$\|v\|_{C(\bar{\Omega})}, \|u_t\|_{C(\bar{\Omega})} < C_1.$$

Now in view of the smoothness of $g(x,t)$, there exists $\lambda > 0$ such that

$$g(x,t) + \lambda t \text{ is nondecreasing on } [-C_1, +C_1], \text{ for all } x \text{ in } \bar{\Omega}.$$

Obviously u is a solution of (1) if and only if u is a fixed point of the compact operator K defined on $C(\bar{\Omega})$ by: $Kv = u$ is given by

$$\begin{cases} -\Delta u + \lambda u = g(x,v) + \lambda v + t\varphi + f, & \text{in } \bar{\Omega}, u \in W^{2,p}(\Omega) \quad (p < \infty), \\ \frac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega. \end{cases}$$

We first prove that if M is large enough, the degree of $I - K$ on $B_M = \{w \in C(\bar{\Omega}), \|w\|_{C(\bar{\Omega})} < M\}$ (with respect to 0) is well defined and $d(I - K, B_M, 0) = 0$.

In order to prove this, we define a family K_s of compact operators in $C(\bar{\Omega})$ defined by: $K_s v = u_s$ is given by

$$\begin{cases} -\Delta u_s + \lambda u_s = s(g(x,v) + \lambda v + f) + (1-s)(1+v^+ + \lambda v) & \text{in } \Omega, \\ \frac{\partial u_s}{\partial v} = 0 & \text{on } \partial\Omega. \end{cases}$$

The same proof as in Step 2) gives, that all solutions u_s of: $u_s = K_s u_s$ satisfy: $\|u_s\|_{C(\bar{\Omega})} < M$ (indep. of $s \in [0,1]$). We will also assume that $M > C_1$. Thus the degree of $I - K_s$ on B_M is well defined and independent of $s \in [0,1]$:

$$d(I - K, B_M, 0) = d(I - K_0, B_M, 0).$$

Now, if u_0 is a solution of: $u_0 = K_0 u_0$, we have

$$-\Delta u_0 = u_0^+ + 1, \quad \frac{\partial u_0}{\partial v} = 0 \text{ on } \partial\Omega, \quad u_0 \in C^2(\bar{\Omega}),$$

and thus $\int_{\Omega} (1 + u_0^+) dx = 0$, which is impossible; thus there is no fixed point of K_0 and $d(I - K_0, B_M, 0) = 0$.

We then prove that if $\mathcal{O} = \{w \in C(\bar{\Omega}), v < w < u_{t_0} \text{ in } \bar{\Omega}\}$ then $d(I - K, \mathcal{O}, 0)$ is well defined and is equal to +1. Indeed let $\varphi \in \mathcal{O}$, and let us define $\tilde{K}_s v$ on $C(\bar{\Omega})$ by

$$\tilde{K}_s v = s K_s v + (1-s)\varphi, \text{ for } s \in [0,1].$$

Because of the choice of v, u_{t_0} and λ we have obviously:

$$K : \bar{\theta} \mapsto \theta \text{ and thus } \tilde{K}_s : \bar{\theta} \mapsto \theta.$$

This implies that $d(I - \tilde{K}_s, \theta, 0)$ is well defined and independent of $s \in [0, 1]$, therefore we deduce

$$d(I - K, \theta, 0) = d(I - \tilde{K}_s, \theta, 0) = d(I - \tilde{K}_0, \theta, 0).$$

Now $\tilde{K}_0 v$ is constant, equal to φ which belongs to θ , thus

$$d(I - \tilde{K}_0, \theta, 0) = +1.$$

We are now able to conclude: indeed by the above arguments we have

$$d(I - K, B_M - \bar{\theta}, 0) = -1;$$

and this means that (1) has a solution which does not belong to $\bar{\theta}$.

III. The convex case.

We now consider the case where g is convex, more precisely we deal with the following problem:

$$(7) \quad -\Delta u = \varphi(u) + f(x) \text{ in } \Omega, \quad u \in C^2(\bar{\Omega}), \quad u = 0 \text{ on } \partial\Omega,$$

where $f \in C^{0,\alpha}(\bar{\Omega})$ (for some $\alpha \in (0,1)$) and where φ satisfies

$$(8) \quad \varphi \text{ is strictly convex on } \mathbb{R}, \quad \varphi \in C^1(\mathbb{R});$$

$$(9) \quad \lim_{t \rightarrow \infty} \frac{\varphi(t)}{t} < \lambda_1$$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω , with Dirichlet boundary conditions.

It is well-known that if $\lim_{t \rightarrow \infty} \frac{\varphi(t)}{t} < \lambda_1$, then (1) has a unique

solution u for every $f \in C^{0,\alpha}(\bar{\Omega})$. In what follows, we will assume in addition to (8)-(9):

$$(10) \quad \lim_{t \rightarrow \infty} \frac{\varphi(t)}{t} > \lambda_1.$$

We then define K to be the set of functions f in $C^{0,1}(\bar{\Omega})$ such that (7) has at least one solution. In addition we set

$$A_m = \{f \in C^{0,1}(\bar{\Omega}), (7) \text{ has at least two distinct solutions}\}$$

$$A_1 = \{f \in C^{0,1}(\bar{\Omega}), (7) \text{ has exactly one solution}\}$$

obviously $K = A_m \cup A_1$.

Our first result states (setting $X = C^{0,1}(\bar{\Omega})$):

Theorem II.1: Under assumptions (8)-(9)-(10), K is a convex set, unbounded with $K \neq \emptyset$ and $E - K$ is nonempty, unbounded.

Furthermore for all $f \in K$, there exists a minimum solution u of (7) such that the first eigenvalue of the operator $-\Delta - \varphi'(u)$ (with Dirichlet boundary conditions) is nonnegative.

In addition $A_m \subset K$ (and $\partial K = A_1$), and for all $f \in K$ then the first eigenvalue of the operator $-\Delta - \varphi'(u)$ is positive.

Remark II.1: This result may be extended to the case of more general elliptic operators and to more general boundary conditions (in particular Neumann conditions). In addition, we may assume that φ depends on x ((9), (10) being uniform in $x \in \bar{\Omega}$).

Remark II.2: This result is an extension of a result due to H. Berestycki [4], where it is assumed in addition that: $\lim_{t \rightarrow +\infty} \frac{\varphi(t)}{t} < \lambda_2$, where λ_2 is the second eigenvalue of $-\Delta$. However in that special case a more precise description of K may be given: indeed (see [4]) i) K is closed, ii) $A_m^0 = K = \{f \in C^{0,1}(\bar{\Omega}), (7) \text{ has exactly two solutions}\}$. We will see below that if we relax the assumption: $\lim_{t \rightarrow +\infty} \frac{\varphi(t)}{t} < \lambda_2$, then we need some assumption to ensure that $A_m^0 = K$, and that K is closed.

Let us for the moment indicate that in general for f in K (7) may have more than two solutions (even an infinite number of solutions): take $\varphi(u) = \frac{2}{N-2} e^u$ and $f = 0$, $N < 10$ with Ω the unit ball in R^N - see D. D. Joseph and T. S. Lundgren [10]). Furthermore we do not know any other assumption than: $\lim_{t \rightarrow +\infty} \frac{\varphi(t)}{t} < \lambda_2$, to ensure that for f in K , (7) has exactly two solutions.

Remark II.3: For f in K , the minimum solution \underline{u} of (7) depends continuously on f .

To simplify notations, we may assume without loss of generality: $\varphi(0) = 0$.

Before going into the proof, let us give two results which answer the questions raised above (in Remark II.2): (we assume for the sake of simplicity $N \geq 3$).

Theorem II.2: Under assumptions (8)-(9)-(10) and if we assume:

$$(11) \quad \lim_{t \rightarrow +\infty} \{ \phi(t) h(t)^{-2/N} t^{-2} \} = 0, \quad \lim_{t \rightarrow +\infty} \varphi(t) t^{-\frac{N+2}{N-2}} = 0,$$

where $\phi(t) = \int_0^t \varphi(s) ds$ and $h(t) = \frac{1}{2} \varphi(t)t - \phi(t) \geq 0$; then $A_m = \frac{0}{K}$ and thus

$A_1 = \beta K$. In addition, for f in ∂K , the first eigenvalue of the operator

$(-\Delta - \varphi'(u))$ is zero, where u is the corresponding solution of (7).

Remark II.4: Let us give a few examples where the (technical) condition (11) is satisfied:

i) if φ satisfies: $\theta \varphi(t)t - F(t) \geq 0$ for $t \geq t_0$, and for some $\theta \in (0, \frac{1}{2})$ then $h(t) \geq (\frac{1}{2} - \theta) t \varphi(t)$, and if we know that

$$\lim_{t \rightarrow +\infty} \varphi(t) t^{-(N+2)/(N-2)} = 0$$

then $\phi(t) h(t)^{-2/N} t^{-2} \leq C t \varphi(t) t^{-2/N} \varphi(t)^{-2/N} t^{-2} \leq C \frac{\varphi(t)^{\frac{N-2}{N}}}{t^{\frac{N+2}{N}}}$ and thus

(11) is satisfied and soon as we have

$$(12) \quad \left\{ \begin{array}{l} \theta \varphi(t)t - F(t) \geq 0 \text{ for } t \geq t_0 \text{ and for some } \theta \in (0, \frac{1}{2}) \\ \lim_{t \rightarrow +\infty} \varphi(t) t^{-(N+2)/(N-2)} = 0. \end{array} \right.$$

(12) has been introduced by A. Ambrosetti and P. H. Rabinowitz [3], and

contains in particular $\varphi(t) = |t|^p$ for $1 < p < \frac{N+2}{N-2}$.

ii) if φ satisfies: $\lim_{t \rightarrow +\infty} \varphi(t) t^{-N/(N-2)} = 0$, then (11) is satisfied.

Indeed since φ is convex, it is easy to prove that $h(t) \geq \alpha \varphi(t) - C$; and then

$$\phi(t) h(t)^{-2/N} t^{-2} \leq C t \varphi(t) \varphi(t)^{-2/N} t^{-2} = C \frac{\varphi(t)^{\frac{N-2}{N}}}{t^{\frac{N+2}{N}}}.$$

If we consider the particular case $\varphi(t) = |t|^p$ (with $1 < p < \infty$) then (8)-(9)-(10) hold obviously and (11) holds if and only if $p < \frac{N+2}{N-2}$. The following example shows that such a restriction is needed and that $\frac{N+2}{N-2}$ is the critical exponent for A_m^0 to be equal to K .

Example: We assume that Ω is starshaped ($N \geq 3$), $\varphi(t) = |t|^p$ with $p \geq \frac{N+2}{N-2}$, and we take $f = 0$. Then (7) is equivalent to

$$(7') -\Delta u = u^p \text{ in } \Omega, u \geq 0 \text{ in } \Omega, u = 0 \text{ on } \partial\Omega, u \in C^2(\bar{\Omega}).$$

Then in view of the results of S. I. Pohozaev [16], (7') has a unique solution $u \equiv 0$. Thus $0 \in A_1^0$. But by an obvious application of the implicit function theorem, for f in $C^{0,\alpha}(\bar{\Omega})$ small, (7') has still a solution and therefore $f \in K$. Hence $0 \in K^0$.

Finally concerning the question of the closedness of K , let us just indicate that problem is entirely similar to the following problem: let $(0, \lambda^*)$ be the maximal interval such that there exists a solution of (13) $-\Delta u = \lambda(\varphi(u) + f(x)), u \in C^2(\bar{\Omega}), u = 0 \text{ on } \partial\Omega;$ where we assume $f \geq 0, \varphi(0) > 0$; then does there exist a solution of (13) for $\lambda = \lambda^*$? This question is answered in M. G. Crandall and P. H. Rabinowitz [7] (see also F. Mignot and J. P. Puel [14]) and just applying their results and techniques, we obtain:

Proposition II.1: If one of the following conditions is satisfied

$$(14) \quad \left\{ \begin{array}{l} t \varphi'(t) \geq 0 \varphi(t), \text{ for } t \geq t_0 \text{ and for some } 0 > 1, t_0 > 0, \\ \lim_{t \rightarrow +\infty} \varphi(t) t^{-(N+2)/(N-2)} = 0; \end{array} \right.$$

$$(15) \quad \left\{ \begin{array}{l} \varphi(t) = at^m + \psi(t), \text{ for } t \geq 0 \text{ and for some } a > 0, \text{ where } : \text{satisfies} \\ \lim_{t \rightarrow +\infty} \frac{\psi(t)}{t^m} = \lim_{t \rightarrow +\infty} \frac{\psi'(t)}{t^{m-1}} = 0; \end{array} \right.$$

or

$$(16) \quad \left\{ \begin{array}{l} \varphi \text{ is a class } C^2 \text{ and satisfies: } \beta \cdot (t)^2 + \varepsilon(t)\varphi''(t) + \mu(\varepsilon(t)) \geq 0 \\ \text{for } t > t_0 ; \text{ with } 0 < \beta < 2 + \mu + \sqrt{\mu} \text{ and } N < 4 + 2\mu + 4\sqrt{\mu} \end{array} \right.$$

where $t_0 > 0$; then K is closed.

Let us remark that the results of D. D. Joseph and T. S. Lundgren [13] show that these conditions are nearly optimal (see also [7], [14], for examples of nonlinearities φ satisfying (14), or (15), or (16)).

Let us now prove Theorems II.1 and Theorems II.2:

Proof of Theorem II.1: We only prove that $A_m^0 \subset K$, since all the other statements follow directly from the proof of H. Berestycki [4].

Let $f_0 \in A_m^0$, there exist at least a minimum solution of (2) \underline{u} and another distinct solution, say $u > \underline{u}$. Since we have

$$-\Delta(u - \underline{u}) = \varphi(u) - \varphi(\underline{u}) = \left\{ \frac{\varphi(u) - \varphi(\underline{u})}{u - \underline{u}} \right\} (u - \underline{u}) ;$$

this implies that the first eigenvalue of the operator

$-\Delta - \frac{\varphi(u) - \varphi(\underline{u})}{u - \underline{u}}$ (this last function being extended by $\varphi'(\underline{u})$ on $\partial\Omega$) is 0. But since φ is strictly convex, we have

$$\frac{\varphi(u) - \varphi(\underline{u})}{u - \underline{u}} > \varphi'(\underline{u}) \text{ in } \Omega ,$$

therefore the first eigenvalue of the operator $-\Delta - \varphi'(\underline{u})$ is positive. Thus by an obvious application of the implicit function theorem, for f near f_0 in X , (7) has a solution i.e.: $f_0 \in K^0$.

Proof of Theorem 11.1: Let $\varphi \in E_1$, we know (by Theorem 11.1) there exist \underline{u} minimum solution of (7) and that the first eigenvalue of $-\Delta + \varphi'(u)$ is positive. To prove that $\varphi \in A_m$, we just need to show there exists a solution v of

$$(17) \quad \begin{cases} -\Delta v = \varphi(\underline{u}(x)) + \varphi'(\underline{u}'(x)) & \text{in } \Omega, \quad v \in C^2(\bar{\Omega}) \\ v > 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Since φ satisfies (11) and since the first eigenvalue of $-\Delta + \varphi'(u)$ is positive, we may apply the existence results of P. L. Lions [13] to conclude.

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